Polynomials for palindromes of $cf(\sqrt{n}\,)$

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Içara, SC, September 2020

Contents

To Bogdan Miś
who cured many
of mathphobia

1. Choosing right tools

In the beginning there was the discovery of Joseph Louis Lagrange (made public in 1770) that real solutions of second degree equations have periodic continued fractions. And five years later there came the result of Leonhard Paul Euler that the inverse of this fact also holds (if the continued fraction of a real number x is periodic, then the number is a solution of such an equation). Since then flow of results on periodic cf's has never stopped.

The word 'periodic' has rather obvious meaning. Periodic phenomena, natural and conventional (seasons or Saturdays) accompany us all our lives. In sciences they may appear as some types of 'symmetries' and in elementary mathematics they come to students when rational numbers are written in decimal notation. The periods in continued fractions for $\sqrt{n}, n \in \mathbb{N}$, ($n$ not a square itself, for then there is no continued fraction), are usually written in this form

$$[\, x, \overline{a_1, a_2, ..., a_k, 2x}\,]$$

and the part $a_1, a_2, ..., a_k$ is palindromic (it reads the same in both directions).

If the initial part 'x' as well as the ending '$2x$' of the period are left aside, the focus is switched from underlying natural numbers to subsets of $\mathbb{N}$ with the same palindromic part. This is the very **first of elementary tools** used here but then one has to decide how to indicate that a period has length 1.

Well, another way of saying *there is no palindrome* or *the palindrome is empty* is *the palindrome has length* 0. (Yes, 0 is an even natural number.)

When they are oft-repeated, long terms are vexing, so, from now on, the term **k-pal** will be used for a palindrome string of length k.

The sequence of cf's for consecutive natural numbers may bewilder as there is no visible pattern. It may be the reason for so much work done on the analysis of the sequence of period lengths. Putting natural numbers in line is next to automatic but the example of Ulam spiral of 1963 shows that other arrangements may help in searching for interesting patterns.

The formula

$$1+3+5+\ldots+2n-1=n^2$$

was known in ancient Greece and interpreted in geometric spirit (though not in the modern notation) by squares.

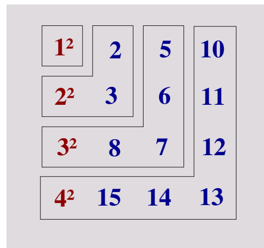

FIGURE 1 – Queue is dull, square better

And many sources affirm that in 1932 Laurence Monroe Klauber layed out $\mathbb{N}$ in a triangle in order to show lines that abound in prime numbers. The triangle used here, the **second elementary tool**, starts with the same idea but the number 0 (and not 1) stands at the top of the triangle.

$$
\begin{array}{c}
0 \\
1^2\ \ 2\ \ 3 \\
2^2\ \ 5\ \ 6\ \ 7\ \ 8 \\
3^2\ \ 10\ \ 11\ \ 12\ \ 13\ \ 14\ \ 15 \\
4^2\ \ 17\ \ 18\ \ 19\ \ 20\ \ 21\ \ 22\ \ 23\ \ 24 \\
5^2\ \ 26\ \ 27\ \ 28\ \ 29\ \ 30\ \ 31\ \ 32\ \ 33\ \ 34\ \ 35 \\
6^2\ \ 37\ \ 38\ \ 39\ \ 40\ \ 41\ \ 42\ \ 43\ \ 44\ \ 45\ \ 46\ \ 47\ \ 48 \\
7^2\ \ 50\ \ 51\ \ 52\ \ 53\ \ 54\ \ 55\ \ 56\ \ 57\ \ 58\ \ 59\ \ 60\ \ 61\ \ 62\ \ 63
\end{array}
$$

FIGURE 2 – Zero entered and L's became rectified

Thus, consecutive rows start with perfect squares, that is: the row number 1 starts with 1 and has 2 more cells for numbers $2,3$, the row number 2 starts with $2^2=4$ and has 4 more cells for numbers $5,6,7,8$, and in general the row number m starts with m^2 and has $2m$ more cells for numbers $\{m^2,\ldots,(m+1)^2-1\}$.

The picture on the next page shows these cells occupied by lengths of the pals of the corresponding square roots. It may be helpful to have an amplied print of the picture as there is no shorter way to intuitions than playing with data just for fun of it. It need not be fast – but it works.

```
1   - 0 1
2   - 0 1 3 1
3   - 0 1 1 4 3 1
4   - 0 1 5 1 5 5 3 1
5   - 0 1 3 4 1 7 3 3 3 1
6   - 0 1 1 1 2 1 9 7 5 11 3 1
7   - 0 1 5 4 5 3 1 5 6 5 3 10 3 1
8   - 0 1 9 1 7 5 7 1 6 4 3 11 5 3 3 1
9   - 0 1 1 4 9 1 5 4 1 7 7 9 15 3 3 10 3 1
10  - 0 1 11 1 1 8 5 7 14 1 5 5 8 5 9 9 5 9 3 1
11  - 0 1 15 4 3 11 3 9 2 5 1 15 13 7 3 8 3 17 3 3 3 1
12  - 0 1 1 1 8 1 19 1 7 9 3 1 16 7 9 7 9 9 17 5 5 21 3 1
13  - 0 1 15 4 3 5 3 7 5 13 3 20 1 5 11 4 9 5 7 3 13 15 3 12 3 1
14  - 0 1 19 1 13 6 1 7 7 7 7 5 7 1 25 13 11 25 3 5 15 4 3 3 5 3 3 1
15  - 0 1 1 4 1 1 5 10 7 1 11 9 7 11 1 16 9 9 25 9 9 11 3 15 6 13 3 21 3 1
16  - 0 1 9 1 15 13 11 1 8 3 5 19 2 5 23 1 5 6 5 11 20 5 7 5 12 7 17 15 5 9 3 1
17  - 0 1 9 4 5 11 5 9 10 3 3 25 15 5 11 3 1 13 5 25 15 15 3 16 6 3 7 10 3 13 3 3 3 1
18  - 0 1 1 1 11 1 33 7 1 27 3 1 18 4 9 13 13 1 15 9 9 4 13 7 6 5 11 7 14 9 11 13 5 25 3 1
19  - 0 1 11 4 11 19 3 11 2 5 5 4 5 9 15 3 7 29 1 5 23 7 7 15 11 5 13 8 3 15 3 19 20 3 3 20 3 1
20  - 0 1 9 1 1 13 5 1 20 1 9 19 7 7 15 7 17 5 17 1 36 13 7 17 6 11 3 11 11 13 15 7 20 3 3 29 5 3 3 1
21  - 0 1 1 4 15 1 1 16 7 9 9 9 33 1 5 24 2 9 11 14 1 31 9 9 21 13 7 9 5 13 13 3 13 9 7 19 35 11 3 12 3 1
22  - 0 1 27 1 21 9 21 7 8 9 1 15 9 5 5 13 27 13 7 3 3 1 5 31 18 5 19 11 15 5 9 9 21 5 17 5 18 9 21 19 5 39 3 1
23  - 0 1 9 4 9 11 9 19 6 7 7 38 15 9 3 5 5 25 13 17 15 5 1 25 10 7 35 4 11 19 3 13 17 9 3 12 17 5 3 18 3 41 3 3 3 1
24  - 0 1 1 1 23 1 9 1 7 22 13 1 39 5 9 1 12 9 9 21 21 9 27 1 30 5 9 43 11 19 23 7 11 8 13 7 32 25 3 9 14 5 37 3 5 19 3 1
25  - 0 1 25 4 1 47 3 17 22 1 11 21 7 15 11 20 5 21 9 9 3 15 5 29 1 5 35 18 15 15 9 17 7 5 5 38 21 3 21 9 15 13 7 37 13 9 3 30 3 1
26  - 0 1 15 1 23 9 13 7 14 15 9 11 1 5 37 13 7 45 7 7 2 8 13 15 4 1 9 7 9 7 7 11 14 7 3 5 13 9 9 27 11 39 27 3 39 13 3 41 5 3 3 1
27  - 0 1 1 4 11 1 7 13 1 45 5 9 13 7 9 13 12 1 11 25 15 51 7 23 8 3 1 10 17 5 9 6 9 21 31 9 43 11 7 36 3 21 29 10 9 11 3 3 12 9 3 17 3 1
28  - 0 1 17 1 23 19 1 1 3 10 3 43 10 1 3 7 19 11 5 9 17 13 13 17 20 7 37 1 5 27 11 7 5 4 9 11 28 5 43 11 13 23 9 5 16 9 17 11 13 21 25 9 5 13 3 1
29  - 0 1 47 4 15 9 5 29 3 11 7 22 7 3 25 16 3 45 3 15 39 19 11 18 7 5 7 27 1 23 9 17 7 15 7 14 15 17 3 24 7 45 7 3 53 11 3 41 3 7 7 21 3 3 3 3 3 1
30  - 0 1 1 1 1 1 41 7 7 1 23 1 31 4 1 13 15 9 59 1 33 18 5 7 4 27 7 15 18 1 21 21 7 13 7 7 32 9 9 7 16 7 5 15 9 29 21 9 26 15 17 7 20 13 15 31 5 35 3 1
31  - 0 1 33 4 5 31 5 15 8 21 19 7 31 3 17 18 9 9 9 29 5 7 9 4 2 3 15 31 3 59 1 5 5 11 19 12 17 17 17 13 19 5 27 17 47 5 3 4 6 7 17 16 15 7 3 23 16 25 3 48 3 1
32  - 0 1 11 1 25 25 27 1 32 5 3 9 8 15 55 1 31 4 13 19 7 25 9 17 24 5 49 15 3 35 3 1 37 15 21 13 34 9 31 7 19 14 3 11 46 13 11 23 6 13 25 5 11 3 19 5 41 12 3 43 5 3 3 1
33  - 0 1 1 4 29 1 9 14 7 27 1 9 9 23 5 4 5 9 37 26 3 1 5 11 32 13 11 40 5 13 7 31 1 29 25 27 9 7 7 4 6 15 31 9 13 31 15 25 12 3 5 57 17 15 13 4 15 23 7 11 15 19 3 42 3 1
34  - 0 1 25 1 25 11 17 7 18 15 5 9 27 5 25 13 1 45 11 5 35 5 5 15 24 19 13 11 17 11 21 5 24 1 5 25 16 21 19 5 11 43 17 7 50 5 9 21 11 9 11 11 17 17 3 15 16 27 3 19 20 3 31 3 5 25 3 1
35  - 0 1 31 4 1 43 1 23 29 1 25 34 9 1 9 14 9 7 27 9 27 7 3 44 12 13 33 9 9 23 9 7 8 25 1 26 23 9 7 9 21 15 25 15 25 9 3 35 17 7 31 20 3 59 7 17 15 25 3 28 17 3 3 34 3 61 3 3 3 1
36  - 0 1 1 1 6 1 51 1 1 24 17 1 29 17 9 7 2 1 13 15 29 49 15 1 44 10 13 63 2 5 7 7 35 13 25 1 27 21 7 31 17 13 11 11 47 7 11 7 5 15 13 41 23 11 5 9 19 26 11 19 37 11 23 9 26 17 9 13 5 69 3 1
37  - 0 1 31 4 19 21 15 17 4 17 5 66 13 9 5 10 7 5 19 9 33 15 3 3 5 7 13 19 13 39 3 39 26 7 7 3 1 5 19 36 5 17 21 29 27 19 13 16 8 5 11 7 15 43 13 3 17 21 7 10 7 5 17 22 27 9 7 27 23 35 3 47 3 1
38  - 0 1 31 1 11 8 33 7 56 19 5 11 21 25 45 13 37 3 1 11 28 14 13 39 7 5 63 5 27 33 11 15 41 5 15 5 36 1 49 5 5 15 15 9 20 4 13 17 18 13 3 5 29 25 45 19 59 19 7 15 13 17 15 3 11 19 35 7 19 12 3 75 5 3 3 1
39  - 0 1 1 4 17 1 19 27 7 57 15 9 1 11 11 19 11 9 5 35 13 51 7 13 6 1 11 68 13 17 13 14 11 31 9 27 21 27 1 59 13 21 27 16 21 11 7 45 16 21 15 9 29 5 41 25 15 69 3 13 13 31 3 24 8 7 49 43 3 31 3 7 24 3 3 48 3 1
40  - 0 1 19 1 1 17 27 1 56 1 9 19 2 23 9 1 13 8 41 1 78 25 5 13 10 29 5 5 37 17 9 7 41 13 9 41 20 3 9 1 5 28 7 15 25 7 5 27 18 11 35 11 17 5 7 9 40 10 13 35 39 15 47 15 15 5 21 17 52 13 9 5 25 23 29 35 5 31 3 1
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The picture brings some hope that it is possible to search for patterns in the flood of computer output.

Some geographic terms will help to deal with these data. The names of **sw-diagonals** and **se-diagonals** – lines running through data downwards in south-west and south-east directions should be easy to accept. As the numbers marked by hyphens – perfect squares – do not lead to any cf's, the sw-diagonal of hyphens gets no number. The next, 'first' sw-diagonal contains zeroes and it is well known that the numbers of the form $n^2 + 1$ have period 1:

$$cf(\sqrt{n^2+1}) = [\, n, \overline{2n}\,]$$

– in fact, the case of n=1 embellishes quite a few introductions to the topic. For some reason $\sqrt{17}$ is not that popular, although the notation $cf(\sqrt{17}) = [\,4, 8, 8, 8, 8, 8, \ldots]$ is also very impressive.

At first glance, three lines of 1's (the first se-diagonal, the central vertical line, the second sw-diagonal) seem meaningful. However, they just say that an apple is an apple is an apple. It is a well-known observation on periodic cf's that

$$cf(\sqrt{m^2+a}) = [\, m, \overline{k, 2m}\,] \text{ if and only if } a \cdot k = 2m$$

So, the three 1's contained in each row (when read from right to left) say: *' 2m divides 2m, m divides 2m, 2 divides 2m'*. (The second se-diagonal composed of 3's is more interesting, and it is connected with neighbouring 1's.) In other cases, number 1 appearing in k^{th} sw-diagonal in m^{th} row ($k < m$) says: $k|m$ (k divides m), so m is not prime. Alas, any primality test based on it does not seem viable.

In order to simplify dealing with number if its cell is placed in m^{th} row and n^{th} sw-diagonal, the symbol $m''n$ will be used

$$m''n = m^2 + n$$

Multiplication by squares has simple notation here:

$$k^2(m''n) = (km)''(k^2n)$$

but the key reason to propose it is that it proves handy in producing the polynomials announced in the title.

2. Enter polynomials

A $(2k+1)$-pal as well as a $2k$-pal depend on k parameters and neighbouring cells of the same row may have the same length of their pals but there will be no connection between their parameters. Consider just the case of numbers $19, 21, 22$ of the 4^{th} row ; their 5-pals are

$$2,1,3,1,2 \qquad 1,1,2,1,1 \qquad 1,2,4,2,1$$

and the more you look at it the less you see anything worth attention.

Before going on, one more convention on shorter notation of palindromes. It is hard to shorten the empty set, so the 0-pal will be denoted by $\ll\ \gg$.

For 1-pal 'k' the symbol is $\ll k \gg$ (although it makes the notation longer; this is the cost of making symbols homogeneous), $2k$-pal $a_1, a_2, ..., a_k, a_k ..., a_2, a_1$ will be written as $\ll a_1, a_2, ..., a_k|$ (the sign '|' symbolizing a mirror) and $(2k+1)$-pal $a_1, a_2, ..., a_k, a_{k+1}, a_k ..., a_2, a_1$ will become $\ll a_1, a_2, ..., a_k|a_{k+1}|$.

For 3-pals or 5-pals the convention may seem a bit useless but one might change mind thinking of $12''7 = 151$ that yields a 19-pal

$$3,2,7,1,3,4,1,1,1,11,1,1,1,4,3,1,7,2,3\ .$$

Or taking the number $41''45$ that gives a 87-pal.

Matricial notation of pari-gp will be often used here: $[r,s]$ for row vector, $[r;s]$ for column vector, $[r,s;t,u]$ for $\left(\begin{smallmatrix} r & s \\ t & u \end{smallmatrix}\right)$ and $S\tilde{}$ for transposition of matrix S.

Going back to 5-pals of the 4^{th} row, in the figure on p.3 there is no other cell that has 5-pal $\ll 2,1|3|$. The figure would need 3 more rows for it: the next number with this pal is $43''31$. The 5-pal of $n = 22$ is a bit closer, at $25''35$ but not close enough for the first encounter with the phenomenon. And the case of 21 is good, for the next position of the same 5-pal is $10''12$.

Movements in the triangle of data can be described by vector $[v,w]$ with v indicating how many rows to descend, and w – how many sw-diagonals to cross going to the right. The move from $4''5$ to $10''12$ is $(6,7)$, so x moves will correspond to the vector $(6x, 7x)$. Moving x times from $4''5$ leads to position $(6x+4)''(7x+5)$. And that is the **pal polynomial** of the 5-pal $\ll 1,1|2|$:

$$(6x+4)^2+(7x+5)=(3^2x+7)(2^2x+3)$$

The move (v,w) is always with $w\geqslant 0$ (going to the left would cross the left border of the triangle in some steps) and with $w\leqslant 2v$ (to prevent such crossing through the right border). So, it looks like an inclined L with not too long base. And the leading coefficient of obtained quadratic polynomial is v^2.

Shouldn't there be a similar reasoning when $7''6$ with 3-pal $\ll 2|2|$ is taken ? The next case of this pal is at $13''11$, so the move is $[6,5]$ but there is something strange with the polynomial

$$(6x+7)^2+(5x+6)=(3^2x+11)(2^2x+5))$$

Both free coefficient in linear factors are too big. But if the first move is made backwards, it leads from $7''6$ to $1''1=2$ where its 0-pal is $cf(\sqrt{2})=[\,1,\overline{2}]$. There is no impediment to treat infinite sequence of 2's as blocks $\ll 2,2,2\gg$ separated by number 2. Due to this change, the polynomial starting at $1''1$ is

$$(6x+1)^2+(5x+1)=(3^2x+2)(2^2x+1)$$

Accepting such type of formally admissible pattern is technically useful and Bézout's identity invoked in the process of forming pal polynomials frequently makes them appear as $cf(\sqrt{f(0)})$ for the obtained polynomial $f(x)$. This is the reason to introduce the concept of two distinct types of roots. As values of polynomials for increasing arguments x are directed downwards in the pal triangle, it is contrary to intuitions to have *roots* at the top but it is common in trees considered in computer sciences, so this convention will be followed here.

A palindromic string s yielding pal polynomial $f(x)$ will appear in $cf(\sqrt{f(x)})$ for all $x\in\mathbb{N}$ but it may happen that in case of $c=cf(\sqrt{f(0)})$ although pal for c is shorter. It is so when $cf(\sqrt{c})=[\,x,a_1,a_2,...,a_k,2x,a_1,a_2,...,a_k,2x,a_1,a_2,...,a_k,2x,...\,]$ and s is longer than k (it coincides with part of the string that starts with a_1 but it includes also $2x$ and some other terms). In such situation c will be called the **illusive root** of $f(x)$. And if s coincides with the pal of c (it is exactly the string between x and $2s$) then c will be called the **staple root** of pal polynomial $f(x)$.

It is easy to make a mistake assuming that in the first case the polynomial $g(x)$ that corresponds to the pal of c must have its staple root at c. In fact, c need not be the initial value of $g(x)$. A simple example may elucidate this: for 5-pal $\ll 2,4|2|$ the pal polynomial is $f(x) = (11^2x+3)(9^2x+2)$ Then $c = 6$ and $cf(\sqrt{6}) = [2,2,4,2,4,2,4,2,4...]$, so here $x = 2$, $2x = 4$ and the pal of c is $\ll 2 \gg$. But the polynomial for this pal is $g(x) = (x+1)x$ and its first value (at $x = 1$) is 2. Therefore 2 is the staple root of $g(x)$.

Number 19 from the beginning of this section is an example of staple root of $(13^2x+19)(3^2x+1)$ (its 5-pal is $\ll 2,1|3|$) and illusive root of $(170^2x+19)(39^2x+1)$ (with its 11-pal $\ll 2,1,3,1,2|8|$).

The need of this distinction is the consequence of passing from periods of continued fractions to their palindromes. If the first emergence of palindromic string $a_1,...,a_m$ is preceded by x and followed by $2x$, there is no ambiguity.

And now there comes the **third elementary tool** which soon proves its value: why not consider negative arguments of x?

This action connects two types of pals: k-pal with $k > 1$ and starting with $a_1 > 1$ is carried over into $(k+2)$-pal that has $1, a_1{-}1$ for its first elements; and vice versa, if $k > 3$ and its two initial elements are $1, a_2$ then negative arguments of its pal polynomial turn it into $(k-2)$-pal that starts with $a_2{+}1$.

The relation between 3-pals $\ll 1|a_2|$ and 1-pals is slightly different and will be explained later. And the relation of the two pal polynomials obtained for the 5-pal $\ll 1,1|2|$ and the 3-pal $\ll 2|2|$ is visualized in the next figure that uses two copies of the triangle forming an infinite hourglass. Its important feature is the inclusion of the mirror image of 'zero line'. [1]

Finding other examples of such construction with small vectors (v,u) is challenging but such a search for patterns is a fresh and gratifying exercise. In order to facilitate this task, the last page contains the figure of the inverted data triangle. With the help of printer, glue, ruler and pari gp one can make some small discoveries. Or not so small ones.

1. It led Igor da Silva Solecki to observe that there may exist two zeroes, one positive and one negative.

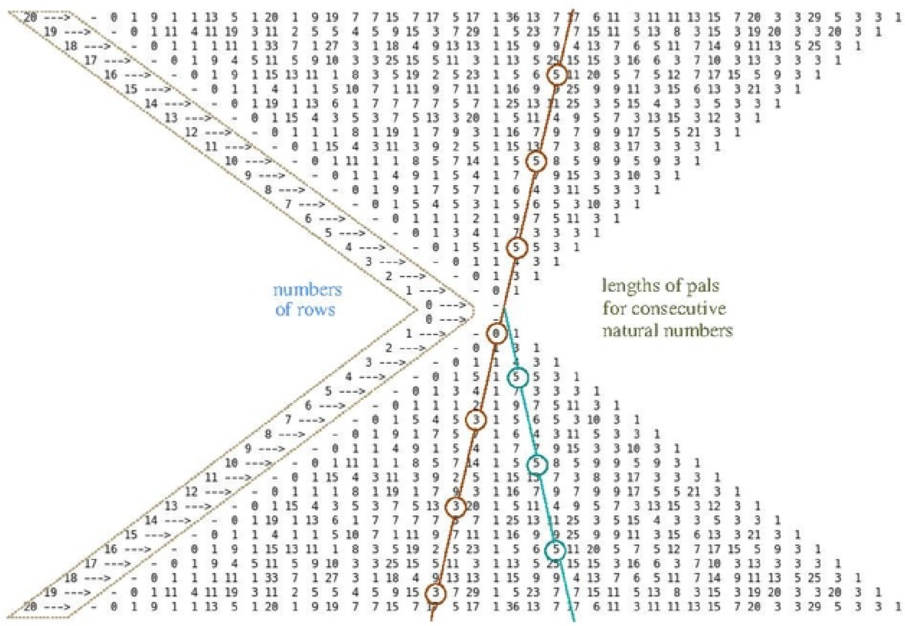

FIGURE 3 – Continued fractions hourglass

3. 0, 1 and 2 are very small numbers

On the page 4 there is already a hint on the polynomial of the 0-pal. It is x^2+1. Indecomponible. Discriminant -4. Its (invisible) leading coefficient is a perfect square. The harbinger of pal polynomials for all $2k$-pals. So much on zero.

Pal polynomials of 1-pals are of two types, one for ≪$2a$≫ and another for≪$2a$–1≫ . In both cases polynomials start at $x = 1$.

The polynomials for even terms are

$$(a^2x+1)x$$

and the ones for odd terms are

$$((2a-1)^2x + 2)x$$

Is the notion of pal polynomial well defined ? No problem with the existence, but what about the uniqueness ? And why not to write $a^2x^2 + x$ and $(2a - 1)^2x^2 + 2x$? These are legitimate questions and they have to be addressed now.

The problem of uniqueness means: if three numbers with the same k-pal are given, with no other numbers of this pal between them, and the polynomials are formed twice, once for the first pair and then for the next one, can one be sure that the change of variable from x to $x+1$ will carry the first polynomial into the other?

Yes, one can. Still, when the algorithms to form the polynomials are given, the question will be not solved but dissolved.

For the other questions it will be helpful to see some examples of pal polynomials. The case of $(3^2x+2)(2^2x+1)$ for 5-pal $\lll 1,1|2|$ and $(3^2x+2)(2^2x+1)$ for 3-pal $\lll 2|2|$ share one property: the central elements of their pals are even. So, it might be good moment to introduce the polynomial of 5-pal $\lll 2,1|3|$ that was dismissed before. The pertinent numbers are $4''3, 43''31$ and the polynomial is

$$(13^2x+19)(3^2x+1)$$

An example of a 3-pal with odd central term would also be welcome, so be it the one obtained from the pair $13''7,\ 28''15$. The 3-pal is $\lll 3|1|$ and its polynomial is

$$(5^2x+22)(3^2x+8)$$

Four examples of odd-length pals obtained so far suggest that restating the informations on 1-pals is important as it reveals their influence on other odd-length pals. Two algorithms of section 5 will show that using a (and not $2a$) in case of $\lll 2a \ggg$ influences $2k+1$-pals with even central term – and that determinant ± 2 of matrix obtained for 1-pal $\lll 2a-1 \ggg$ will be shared by all $2k+1$-pals with odd central terms.

1-pal $\lll 2a \ggg$	3-pal $\lll 2\|2\|$	5-pal $\lll 1,1\|2\|$	1-pal $\lll 2a-1 \ggg$	3-pal $\lll 3\|1\|$	5-pal $\lll 2,1\|3\|$
$(a^2x+1)x$	$(3^2x+2)(2^2x+1)$	$(3^2x+7)(2^2x+3)$	$((2a-1)^2x+2)x$	$(5^2x+22)(3^2x+8)$	$(13^2x+19)(3^2x+1)$
$\begin{pmatrix} a^2 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 3^2 & 2 \\ 2^2 & 1 \end{pmatrix}$	$\begin{pmatrix} 3^2 & 7 \\ 2^2 & 3 \end{pmatrix}$	$\begin{pmatrix} (2a-1)^2 & 2 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 5^2 & 22 \\ 3^2 & 8 \end{pmatrix}$	$\begin{pmatrix} 13^2 & 19 \\ 3^2 & 1 \end{pmatrix}$
det $= -1$	det $= 1$	det $= 1$	det $= -2$	det $= 2$	det $= -2$

Passing from polynomial notation of two linear factors to 2×2 matrix of coefficients and the reverse passage are elementary for humans with pen and paper. The matrix notation is useful in work with computer but going back to polynomial one in this case asks for some gimmicks: if S is the matrix of coefficients such as exposed in the table, the polynomial is recovered by the multiplication

$$(S[x;1])\tilde{}[0,1/2;1/2,0](S[x;1])$$

– or nearly recovered, as it becomes a polynomial closed in the form of 1×1 matrix. Good news is that in the case of even-length pals the process will be simpler.

The observation on p.7 called 'the third elementary tool' deserves some more space; explaining some fine points will make the path smooth when formulas for all possible pals will be introduced. (The word 'possible' is used here in both senses, describing pals that actually can occur when natural numbers are considered – and dealing with pals 'as long as one may wish'.)

So, for a given pal polynomial, substitution of x by $-x$ is not desired as negative coefficients would appear – and one would rather avoid it. Besides, the formulas to come use absolute values that appear in Bézout's identity and there is always choice between two pairs of such numbers. One has better recall this trivial detail, so that a simple example may prevent possible misunderstanding. Taking numbers $11, 7$ one gets

$$2 \times 11 + (-3) \times 7 = 1; \qquad (-5) \times 11 + 8 \times 7 = 1$$

and one does not know whether $2, -3$ or $-5, 8$ may be necessary.

The dependence on computer algorithm that furnishes one or another pair results in obtaining one of two possible polynomials. They will be produced by matrix S where column $[r;s]$ consists of relatively prime numbers and column $[t;u]$ of absolute values of numbers from Bézout's identity. Shorter pals are prefered to longer ones. So, if polynomial of the longer pal, starting with 1, appears (such a polynomial will be denoted by $\hat{f}(x)$) and the matrix bringing the solution is $\left(\begin{smallmatrix} r & t \\ s & u \end{smallmatrix}\right)$, then multiplication $\left(\begin{smallmatrix} r & t \\ s & u \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 1 & 1 \\ 0 & -1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} r & r-t \\ s & s-u \end{smallmatrix}\right)$ will give the required polynomial $f(x)$.

And what is the difference in passage from $f(x)$ to $\hat{f}(x)$ when 1-pals and 3-pals are considered? First of all, for $\ll 2a-1 \gg$ when $a=1$, one should not use in the polynomial the argument $x=-1$, so there is a break in two polynomials f and $\hat{f}$ and they turn out to be identical if the change of variable $x \to -x-2$ is applied. Next, in the case of $\ll 2 \gg$, the central line of values in the hourglass of p.8 is preserved in its position, or in other terms, this polynomial does not change under the substitution of variable $x \to -x-1$.

For parameters different from $a=1$, if $f(x)$ corresponds to $\ll 2a-1 \gg$ then $\hat{f}(x)$ corresponds to $\ll 1|2a-3|$. And if $f(x)$ corresponds to $\ll 2a \gg$ then $\hat{f}(x)$ corresponds to $\ll 1|2a-2|$. Line of polynomial of any $\ll k \gg$ passes (only once) in the hourglass of p.8 through the second se-diagonal (which, when reflected, becomes ne-diagonal) and each of its elements $m''(2m-1)$ is met by such a line. It makes clear why this diagonal is composed of 3-pals $\ll 1|m-1|$.

The next section deals with even-length pals but the description of 2-pals is given now, as it makes natural company for easy results of cases $k=0$ and $k=1$. First of all, there are **no** polynomials corresponding to $\ll 2a-1|$ when **natural** numbers are considered. (The explication of this non-existence will come soon.) And the rule giving polynomial for any $\ll 2a|$ is

$$f(x) = (4a^2+1)^2x^2 + 2a(4a^2+3)x + (a^2+1)$$

It becomes even simpler if matricial notation is applied; if $M = \left(\begin{smallmatrix} 4a^2-1 & a \\ 4a & 1 \end{smallmatrix}\right)$, the polynomial is $(M[x;1])\tilde{}\,(M[x;1])$.

The algorithms of two following sections will show how easy it is to get the answer to such questions:

Prime number 379 *corresponds to* 29*-pal* $\ll 2,7,3,2,2,6,12,1,4,1,1,1,3,4|19|$ *and prime number* 541 *corresponds to* 38*-pal* $\ll 3,1,5,1,8,2,4,1,2,3,1,1,11,15,2,2,1,1,1|$. *In both cases, what other numbers have the same pals ?*

When the answer is given by the polynomial

$$(113759383^2x+379)(5843427^2x+1)$$

for the first case, and for the other number by the polynomial

$$58536158470221581^2x^2 + 2\cdot 13615163164692274 50x + 541,$$

there arises the next question: *why two solutions have so different forms ?*

4. Even-length pals

The striking feature of the figure *'40 rows'* on p.3 is the sparsity of even numbers.

All $2k$-pal polynomials have discriminant equal to -4. In other words: coefficients of such a polynomial $v^2x^2+2bx+c$ satisfy the condition $b^2+1 = v^2c$ which is known as *negative Pell equation*. It is a very attractive topic and its connections with pal polynomials will be dealt with in section 6.

4.1. Initial considerations

Coefficient v will be called **descent index** of the polynomial If v is prime and other factors in decomposition of d^2+1 are square-free, then v alone determines the polynomial that (starting at $x=0$) gives all numbers of required pal. Otherwise, more cases are possible; for example, factoring of $81536^2{+}1$ gives $17^3 \cdot 29^2 \cdot 1609$ and possible v's are $17, 29, 493 = 17 \cdot 29$. Here are respective polynomials:

$17^2x^2 + 2 \cdot 38x + 5$	$f(282) = 17{\cdot}29^2{\cdot}1609\,;$	2-pal $\ll 4\|$,
$29^2x^2 + 2 \cdot 800x + 761$	$f(96) = 17^3{\cdot}1609\,;$	6-pal $\ll 1, 1, 2\|$,
$493^2x^2 + 2 \cdot 81356x + 27353$	$f(0) = 17{\cdot}1609$	10-pal $\ll 2, 1, 1, 2, 1\|$,
and another case of $v{=}17{\cdot}29$ *but with* $b{=}9210$,	$9210^2{+}1 = 493^2{\cdot}349$	
$493^2x^2 + 2 \cdot 9210x + 349$	$f(0) = 349\,;$	6-pal $\ll 1, 2, 7\|$.

The list of all possible polynomials can be obtained with formulas known for more that 2300 years and they constitute **fourth of elementary tools**. They are the formulas for Pythagorean triples. If two natural numbers p, q are relatively prime and $p > q$ then the identity

$$(p^2 - q^2)^2 + (2pq)^2 = (p^2 + q^2)^2$$

gives all non-trivial cases of the famous equation $a^2 + b^2 = c^2$ (multiplying elements of $3^2 + 4^2 = 5^2$ by, say, 3, to get $9^2 + 12^2 = 15^2$ is trivial).

This identity will be used together with **fifth tool** – Pierre de Fermat's theorem on sums of two squares, contained in a letter of 1640:

Prime number p can be expressed as $a^2 + b^2$ if and only if it has form $p = 4k{+}1$ for some $k \in \mathbb{N}$.

The notation $r\%s$ of pari-gp meaning '$r \pmod s$' will be used here. So, there must be $p\%4 = 1$.

What about possibilities of taking products of such primes and expressing them also as sums of squares ? Another *very* classic result, called Brahmagupta– Fibonacci identity:

$$(a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2 = (ac + bd)^2 + (ad - bc)$$

shows that it can be done in two manners. This identity is the basis of creation called *complex numbers* – it is astonishing that they were not invented in the antiquity – and using them might make it easier to visualize the procedure.

As $41 = (5+4i)(5-4i)$ and $101 = (10+i)(10-i)$, it is enough to use two pairs of factors :

$$(5 + 4i)(10 + i) = 46 + 45i$$
$$(5 + 4i)(10 - i) = 54 + 35i$$

so that $41 \cdot 101 = 46^2 + 45^2 = 54^2 + 35^2$.

It should be clear why product of three factors will lead to 4 distinct results. For example, if $1105 = 5 \cdot 13 \cdot 17$ is considered, and the factors can be decomposed $5 = (2+i)(2-i),\ 13 = (2+3i)(2-3i),\ 17 = (4+i)(4-i),$ then

$$\begin{array}{ll} (2+i)(2+3i)(4+i) = -4+33i & (+\,+\,+) \\ (2-i)(2+3i)(4+i) = 24+23i & (-\,+\,+) \\ (2+i)(2-3i)(4+i) = 32-9i & (+\,-\,+) \\ (2+i)(2+3i)(4-i) = 12+31i & (+\,+\,-). \end{array}$$

so that

$$1105 \quad = \quad 4^2 + 33^2 \quad = \quad 24^2 + 23^2 \quad = \quad 32^2 + 9^2 \quad = \quad 12^2 + 31^2$$

The following description explains how all $2k$-pal polynomials for $k > 0$ arise (the case $k{=}0$ is settled already by the polynomial $x^2{+}1$).

1. For a pair $p, q \in \mathbb{N}^*$ such that $p > q$, $\gcd(p, q) = 1$, $p\%2 \neq q\%2$, the pair $(a, b) = (p^2 - q^2, 2pq)$ is formed. It will make the first column $[a; b] = [p^2 - q^2; 2pq]$ of matrix S. (The sum of the squares is $v^2 = (p^2 + q^2)^2$.)

2. Conditions on p, q guarantee that $\gcd(p^2 - q^2, 2pq) = 1$, and coefficients of Bézout's identity for the pair $2pq, p^2 - q^2$ (reversed order!) are sought. In pari-gp the name of function is *gcdext*. They form the second column $[t; u]$ of matrix S.

3. Those coefficients (more precisely: the norm $t^2 + u^2$ of the column vector that they make) are used to find out what pal resulted: if the first term of the pal is 1, the correction discussed on p.10 is applied.

4. Each obtained result: v, M and the pal of $cf(\sqrt{t^2 + u^2})$ is colocated in the list containing all previous results sorted by increasing values of v – unless there are doubts because of phenomenon called on p.6 *formally admissible pattern*. In this case it is necessary to see $cf(\sqrt{f(1)}$ (and $f(1)$ is the sum of coefficients of $S\tilde{}S$) to confer which pal is obtained.

There is an amusing effect of the procedure: it starts with Pythagorean triple and ends up with a nearly-Pythagorean triple $(vt)^2 + (vu)^2 = b^2 + 1$.

But there is no need to run over all possible pairs p, q – matrix multiplication does all the work.

4.2. 'Give $2k$-pal, receive polynomial' algorithm

For every $n \in \mathbb{N}^*$ matrix $M(n)$ is defined as:

$$M(n) = \begin{pmatrix} n & 1 \\ 1 & 0 \end{pmatrix}$$

This definition suffices to formulate the algorithm.

1. For given $2k$-pal $\ll a_1, ... a_k|$ make $list = \{a_1, ... a_k\}$ and multiply matrices $M(a_1) \cdot ... \cdot M(a_k)$. If the first row $[r, s]$ of the product consists of two odd numbers, abandon the case (such a pal is impossible for $n \in \mathbb{N}$). Otherwise, go to point 2.

2. Make $[r^2 - s^2; 2rs]$ the first column of matrix $S = SOL(list)$. For the second column take $[t; u]$ – the absolute values of numbers that appear as coefficients of Bézout's identity for the pair $(2rs, r^2 - s^2)$ (reversed order!)

3. Confer if the first term of palindrome in $cf(\sqrt{t^2 + u^2})$ is equal to a_1. If true, form the polynomial $[x, 1]S\,\tilde{}\,S[x; 1]$. Otherwise, substitute S by $\hat{S} = S\left(\begin{smallmatrix}1 & 1\\ 0 & -1\end{smallmatrix}\right)$ and form the polynomial $[x, 1]\hat{S}\,\tilde{}\,\hat{S}[x; 1]$.

4. Leave the field with flying colors.

4.3. $2k$-pals impossible for $cf(\sqrt{n})$ when $n \in \mathbb{N}$

The remark from p.11 that 2-pal $\ll 2a-1|$ is impossible for $n \in \mathbb{N}$ can be easily explained now. The first row $[r, s]$ of $M(2a-1) = \left(\begin{smallmatrix}2a-1 & 1\\ 1 & 0\end{smallmatrix}\right)$ has two odd numbers, so that $\gcd(r^2 - s^2, 2rs)$ is even. But $SOL(list)$ has to have determinant ± 1.

It catches the eye that some $2k$-pals composed of 1's are absent in the next table. It contains the list of first 12 cases with columns describing $2k$, descent index (if it exists), $(M(1))^k$ and the desired $2k$-pals.

2	none	$[1, 1; 1, 0]$	$\ll 1|$
4	5	$[2, 1; 1, 1]$	$\ll 1, 1|$
6	13	$[3, 2; 2, 1]$	$\ll 1, 1, 1|$
8	none	$[5, 3; 3, 2]$	$\ll 1, 1, 1, 1|$
10	89	$[8, 5; 5, 3]$	$\ll 1, 1, 1, 1, 1|$
12	233	$[13, 8; 8, 5]$	$\ll 1, 1, 1, 1, 1, 1|$
14	none	$[21, 13; 13, 8]$	$\ll 1, 1, 1, 1, 1, 1, 1|$
16	1597	$[34, 21; 21, 13]$	$\ll 1, 1, 1, 1, 1, 1, 1, 1|$
18	4181	$[55, 34; 34, 21]$	$\ll 1, 1, 1, 1, 1, 1, 1, 1, 1|$
20	none	$[89, 55; 55, 34]$	$\ll 1, 1, 1, 1, 1, 1, 1, 1, 1, 1|$
22	28657	$[144, 89; 89, 55]$	$\ll 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1|$
24	75025	$[233, 144; 144, 89]$	$\ll 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1|$

If $a_1 = ... = a_k = 1$ then vectors $[r, s]$ contain consecutive Fibonacci numbers and starting with their first coupling $[1, 1]$ every third pair consists of two odd numbers. Descent indices that stand in the second column of the table form a sequence that has index *A190949* in the On-Line Encyclopedia of Integer Sequences and is titled *Odd Fibonacci numbers with odd index*.

As a by-product of the above observation there comes the conclusion that for $k\%3 = 1$ any $2k$-pal must contain some even elements.

On the other hand, a look on the product $M(2r)M(2s)$ shows that first row of product of any number of matrices M depending on even arguments cannot have first row of two odd numbers. In other word, for any $2k$-pal that is a string of even numbers there exist $n \in \mathbb{N}$ with $cf(\sqrt{n})$ having exactly this pal.

So, it may be interesting to look at another $2k$-pals with equal terms, all of them being 2. The sequence of matrices $(M(2))^k$ starts like this:

$$\begin{array}{llll} [2,1;1,0] & [5,2;2,1] & [12,5;5,2] & [29,12;12,5] \\ [70,29;29,12] & [169,70;70,29] & [408,169;169,70] & [985,408;408,169] \end{array}$$

and one recognizes the elements of the sequence $0, 1, 2, 5, 12, 29, 70, 169, 408, 985, \ldots$ as *Pell numbers* (*A000129* in OEIS) – the neighboring pairs taken as q, p in the process of forming Pythagorean triples give such triples that differences of obtained $p^2 - q^2$ and $2pq$ are ± 1.

What types of even-length pal strings cannot appear as $cf(\sqrt{n})$? And how many of these types are there for given $2k$? Two examples above suggest that viability of the string involves only its parity type and recursive process of construction.

The categories *good* and *bad* do not make sense for pals but the words are short; therefore they will be used here. So, *bad*(k) and *good*(k) will express impossible and viable types of $2k$-pals, respectively.

Counting of good cases starts with numbers 1 (for empty pal) and 1 for $k = 1$. And the general rule for consecutive lengths is:

$$\#\mathbf{bad(k+1)} = \#\mathbf{good(k)}$$

Both types sum up to 2^k for each length (there are no bad cases for $k = 0$, so the rule is valid in this case, too, as $2^0 = 1$) and $1, 1, 3, 5, 11, 21, 43, 85, \ldots$ is known as **Jacobsthal sequence** (*A001045* in OEIS).

Here is the proof of the rule. When the viability of strings is concerned, it suffices to consider matrix $S\%2$, and pairs $(r\%2, s\%2)$ in their rows are of three types: $(0,1)$, $(1,0)$, $(1,1)$. As $\det(M(n) = -1$ and $\det(AB) = (\det(A)) \cdot (\det(B))$, all products of $M(n)$'s have determinants ± 1 – and a row of type $(0,0)$ would imply that determinant is at least 2.

Thus, there are four possibilities for $S\%2$, where S is matrix leading to a viable $2k$-pal (its first row cannot be $[1,1]$):

$$S_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad S_2 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \qquad S_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad S_4 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

When multiplied on the left by $M(1)$ and $M(2)$, the first rows of obtained matrices determine whether they serve to create $2(k+1)$-pals. Of all eight possible products, four types of results

$$M(2)\cdot S_1 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \; M(1)\cdot S_2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \; M(2)\cdot S_3 = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}, \; M(1)\cdot S_4 = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$$

are good, other four are bad. It means that every good $2k$ type leads to exactly one bad among $2k+1$ zero-one strings.

To complete the proof it is necessary to check by analogous reasoning that when multiplied on the left by any matrix $M(n)$, bad $2k$ cases make always good $2(k+1)$ ones, so there is no supply of bad $2(k+1)$ cases apart from the ones that have been already discussed.

A short list of all possible zero-one strings for $1 \leqslant k \leqslant 6$ enters the following small table, where the letter **B** indicates that the string on the left is bad.

k=1	**k=4**	**k=5**		**k=6**			
0	0000	00000	10000 **B**	000000	010000	100000 **B**	110000
1 **B**	0001	00001 **B**	10001	000001	010001	100001	110001 **B**
k=2	0010 **B**	00010	10010	000010 **B**	010010 **B**	100010	110010
00	0011	00011 **B**	10011	000011	010011	100011	110011 **B**
01	0100	00100 **B**	10100	000100	010100	100100	110100 **B**
10 **B**	0101	00101	10101 **B**	000101	010101	100101 **B**	110101
11	0110 **B**	00110	10110	000110 **B**	010110 **B**	100110	110110
k=3	0111	00111	10111 **B**	000111	010111	100111 **B**	110111
000	1000 **B**	01000	11000	001000 **B**	011000 **B**	101000	111000
001 **B**	1001	01001 **B**	11001	001001	011001	101001	111001 **B**
010	1010	01010	11010 **B**	001010	011010	101010 **B**	111010
011 **B**	1011	01011 **B**	11011	001011	011011	101011	111011 **B**
100 **B**	1100	01100 **B**	11100	001100	011100	101100	111100 **B**
101	1101 **B**	01101	11101	001101 **B**	011101 **B**	101101	111101
110	1110	01110	11110 **B**	001110	011110	101110 **B**	111110
111	1111 **B**	01111	11111	001111 **B**	011111 **B**	101111	111111

Some regularities in positioning of **B**'s are explained when 'good goes to bad' rule is connected to the passage from f to $\hat{f}$. There is a change of the first component of a pal – a_1 passes to $1, a_1-1$, provided that $a_1 \neq 1$, so that for $a_1\%2$ there is the substitution of 0 by 11 – and 1 by 10. When no f exists, there cannot be $\hat{f}$ either. So, bad type of string that begins with 0 gives rise to longer bad one with 0 substituted by 11 and bad one beginning with 1 – to longer bad one with 10 in place of 1.

4.4. $2k$-pals and form of p, q

The table below retells results of the algorithm for $k=1,2,3,4,5$. (Matrizes $M(n)$ are not mentioned here.) It becomes visible how $2k$-pal is reflected in the structure of pair p, q that was introduced on p.14.

The chosen string $\{a_1, a_1, \ldots, a_k\}$ has to comply with restrictions on parity type of pals that has been discussed in the previous subsection.

$2k$-pal	p, q
$\ll a_1\|$	$a_1, 1$
$\ll a_1, a_2\|$	a_2a_1+1, a_1
$\ll a_1, a_2, a_3\|$	$(a_3a_2+1)a_1+a_3, a_2a_1+1$
$\ll a_1, a_2, a_3, a_4\|$	$((a_4a_3+1)a_2+a_4)a_1+(a_4a_3+1), (a_3a_2+1)a_1+a_3$
$\ll a_1, a_2, a_3, a_4, a_5\|$	$(((a_5a_4+1)a_3+a_5)a_2+(a_5a_4+1))a_1+((a_5a_4+1)a_3+a_5),$ $((a_4a_3+1)a_2+a_4)a_1+(a_4a_3+1)$

4.5. Going back to number 4

The problem is solved for all possible $2k$, does it make sense to dwell on $k = 2$? It might, if another approach brought other informations.

Two sets of coefficients for all possible 4-pal polynomials will be given. One of them covers the case $\ll 2i, 2j|$, the other one corresponds to all the other cases when the second element of the string is odd.

If the pal is $\ll 2i, 2j|$, its polynomial has coefficients

$$f(x, i, j) = v_{i,j}^2 x^2 + 2b_{i,j}x + c_{i,j}$$

that are expressed in terms of $w_{i,j} = 4ij + 1$ and $z_{i,j} = w_{i,j}j + i$, as follows:

$$v_{i,j} = w_{i,j}^2 + 4i^2, \quad b_{i,j} = 4z_{i,j}^3 + 3z_{i,j}, \quad c_{i,j} = w_{j,j}^2(z_{i,j}^2 + 1)$$

They are similar to coefficients for 2-pal $\ll 2i|$ from p.11:

$$\text{if} \qquad f(x, i)) = v_i^2 x^2 + 2b_i x + c_i$$

$$\text{then} \qquad v_i = 4i^2 + 1, \quad b_i = 4i^3 + 3i, \quad c_i = i^2 + 1$$

and it suggests that a set of closed-form expressions for polynomial coefficients might be found for $k > 2$, too.

For the case of $\ll i, 2j-1|$ other formulas for $w_{i,j}$ and $z_{i,j}$ are necessary: $w_j = 2j^2 - 2j + 1$ and $z_{i,j} = 2w_j i + 2j - 1$. Then

$$v_{i,j} = z_{i,j}i + 2ij - i + 1, \quad b_{i,j} = \frac{z_{i,j}^3 + 3z_{i,j}}{2}, \quad c_{i,j} = w_j^2(z_{i,j}^2 + 4)$$

In both cases direct verification confirms that

$$b_{i,j}^2 + 1 = v_{i,j}^2 \cdot c_{i,j} \ .$$

Straightforward listing of sets of coefficients of $f(x,i,j)$ in a chart (they might be obtained multiplying $M(i)M(j)$ for the viable cases of $\ll i,j|$) does not show many regularities and does not bring much insight how to get such a description for longer even-length pals. On the other hand, there is a hidden cost of using the pattern presented here: these pal polynomials do not start at the smallest possible n with the desired 4-pal $\ll i,j|$. Still, it will happen after the change of variable from x to $x - corr(i,j)$ where $corr(i,j) = \lfloor \sqrt{c_{i,j}}/v_{i,j} \rfloor$.

The next section, dealing with odd-length pals, also contains a part where only a specific case, that of number 3, is examined. The rationale for exposition of these cases is their apparent independence on coefficients of Bézout's identity. In the formulas presented here only addition and multiplication are used for 4-pals, and these operations with some reductions $\mod m$ for 3-pals – and there is no sign of recursive process of long division that leads to Bézout's identity. However, processes of forming the polynomials via matrices $M(n)$ manifestly rely on this identity.

Three possibilities come to mind, perhaps one of them (or none) is correct.

1. Apart from using Bézout's identity, there are other ways to produce the second column of S so that $\det(S) = \pm 1$.

2. This achievement is due to specific types of numbers that arise in S.

3. The recursive process is in some way hidden in the coefficients of $f(x,i,j)$.

What is really going on here? It may be a hard question. Or an easy one.

There is one more reason to re-examine the cases of pals with $k = 2,3,4$. It may be a long shot but when one sees an interplay of variables in powers 2 and 3 a suspicion appears that there is a playing field for methods of algebraic geometry.

5. Odd-length pals

The form of $(2k-1)$-polynomials, alluded to at the end of section 3, is

$$(r^2x + t)(s^2x + u)$$

The case of $u = 0$ brings 1-pal polynomials already described on p.8 and repeated now for the sake of completeness. They are

$$(j^2x+1)x \quad \text{for even term} \ll 2j \gg,$$

$$((2j-1)^2x+2)x \quad \text{for odd one} \ll 2j-1 \gg.$$

They do not conform with formation rules described in two algorithms given in next subsections.

5.1. Even central term in odd-length pal string

The algorithm introduced here uses the matrices $M(n)$ defined in section 4.

0. For the 3-pal $\ll 1|2m|$ the polynomial is $((2m+1)^2x+(2m+1)^2-1)(x+1)$.

1. For given $(2k+1)$-pal $\ll a_1, ...a_k|2m|$, $k, m > 0$, make $list = \{a_1, ...a_k, 2m\}$ and form matrix $M(a_1) \cdot ... \cdot M(a_k) \cdot M(m)$ (it is m, not $2m$ in the last matrix!) The product has determinant ± 1, so the elements r, s of its first rows must satisfy $\gcd(r, s) = 1$ and no type of initial string is rejected. So, go to point 2.

2. Make $[r^2; s^2]$ the first column of matrix $S = SOL(list)$. For the second column take $[t; u]$ – the absolute values of numbers that appear as coefficients of Bézout's identity for the pair (s^2, r^2) (reversed order!)

3. Confer if the first term of palindrome in $cf(\sqrt{tu})$ is equal to a_1. If true, multiply S by $[x; 1]$ and form the polynomial as product of its rows. Otherwise, substitute S by $\hat{S} = S\left(\begin{smallmatrix} 1 & 1 \\ 0 & -1 \end{smallmatrix}\right)$, multiply $\hat{S}[x; 1]$ and then form product of its rows.

5.2. Odd central term in odd-length pal string

Matrix $\quad N(2m-1) = \begin{pmatrix} 2m-1 & 1 \\ 2 & 0 \end{pmatrix} \quad$ will be used in this algorithm.

0. For the 3-pal $\ll 1|2m-1|$ the polynomial is $((2m+1)^2x+(2m+1)^2-2)(x+1)$.

1. For given $(2k+1)$-pal $\ll a_1, ...a_k|2m-1|$, $k, m > 0$, make $list\{a_1, ...a_k, 2m-1\}$ and form matrix $M(a_1) \cdot ... \cdot M(a_k) \cdot N(2m-1)$. If both elements of the first row $[r, s]$ of the product are even, that pal is impossible for $n \in \mathbb{N}$. Otherwise, go to point 2.

2. Make $[r^2; s^2]$ the first column of matrix $S = SOL(list)$. For the second column take $[2t; 2u]$ – twice the absolute values of numbers that appear as coefficients of Bézout's identity for the pair (s^2, r^2) (reversed order !)

3. Confer if the first term of palindrome in $cf(\sqrt{2t \cdot 2u})$ is equal to a_1. If true, multiply S by $[x; 1]$ and form the polynomial as product of its rows. Otherwise, substitute S by $\hat{S} = S\left(\begin{smallmatrix} 1 & 1 \\ 0 & -1 \end{smallmatrix}\right)$, multiply $\hat{S}[x; 1]$ and then form product of its rows.

It might be the right moment to remark that odd-length pal may have illusive root in number that is staple root of even-length polynomial. For example 9-pal $\ll 1, 1, 1, 1|6|$ gives polynomial $f(x) = (5^2x+1)(18^2x+13)$, its illusive root is 13 and this number is the staple root of polynomial $g(x) = 5^2x^2+2\cdot 18x+13$ corresponding to 4-pal $\ll 1, 1|$.

5.3. Impossible types of odd-length pal string

Impediments come only in case of odd central terms and Jacobsthal sequence appears again when one counts impossible types of pals. The analysis is similar to that of pp.15-16.

If the product M of matrices $M(n)$'s is multiplied by $N(2m-1)$ giving both r, s even in $[r, s]$ then $M\%2$ has first row of type $[0, 1]$. Thus, possibilities for the parity type of $M\%2$ are $[0, 1; 1, 0]$ and $[0, 1; 1, 1]$. For $2k-1=1$ it is $M(2)\%2$, for $2k-1=2$ it is $M(1)^2\%2$, and then there goes the repetition of the reasoning that 'good' types multiplied on the left by either $M(1)$ or $M(2)$ turn to be 'bad' ones, but not in both multiplications.

So, impossible type of string for 3-pals is $0|1|0$, for 5 – $11|1|11$ and for 7-pals there are three impossible types: $000|1|000$, $010|1|010$, $101|1|101$.

The next table, listing the impossible cases for $k = 3, 5, 7, 9, 11, 13$ is analogous to that of p.18. Remarks following that table (referring to regularities of positions of letter **B**) are valid here, too. The context makes it clear that any $0-1$ *string* of the table describes a $2k+1$-pal of type '*string*|1|*inverted string*'.

k=1	**k=4**	**k=5**		**k=6**			
0 **B**	0000	00000 **B**	10000	000000	010000	100000	110000 **B**
1	0001	00001	10001 **B**	000001	010001	100001 **B**	110001
k=2	0010	00010 **B**	10010	000010	010010	100010	110010 **B**
00	0011 **B**	00011	10011	000011 **B**	010011 **B**	100011	110011
01	0100	00100	10100 **B**	000100	010100	100100 **B**	110100
10	0101	00101 **B**	10101	000101	010101	100101	110101 **B**
11 **B**	0110	00110	10110 **B**	000110	010110	100110 **B**	110110
k=3	0111 **B**	00111	10111	000111 **B**	010111 **B**	100111	110111
000 **B**	1000	01000 **B**	11000	001000	011000	101000	111000 **B**
001	1001 **B**	01001	11001	001001 **B**	011001 **B**	101001	111001
010 **B**	1010	01010 **B**	11010	001010	011010	101010	111010 **B**
011	1011	01011	11011 **B**	001011	011011	101011 **B**	111011
100	1100 **B**	01100	11100	001100 **B**	011100 **B**	101100	111100
101 **B**	1101	01101 **B**	11101	001101	011101	101101	111101 **B**
110	1110 **B**	01110	11110	001110 **B**	011110 **B**	101110	111110
111	1111	01111	11111 **B**	001111	011111	101111 **B**	111111

When two tables are superimposed it becomes conspicuous that marks of 'bad cases' do not coincide.

5.4. Going back to number 3

There are three sets of formulas that express coefficients of 3-pal polynomials – one for each of three types of numbers in 3-pals, namely

000: even – even – even ;

101: odd – even – odd ;

111: odd – odd – odd.

Case 000.

$$f(x,r,s) = ((2r)^2x+t_{r,s})((2rs+1)^2x+u_{r,s})$$

$$t_{r,s} = (2r-1)^2-(4r(s-1))\%((2r)^2)$$

$$u_{r,s} = ((2rs)^2 - 4rs(s^2-1) - (3s^2-1))\%((2rs+1)^2)\ .$$

Case 101.

$$f(x,r,s) = ((2r-1)^2x+t_{r,s})(((2rs-1)s+1)^2x+u_{r,s})$$

$$t_{1,u} = 1;\ \text{if } r>1 \text{ then }\ t_{r,s} = ((2r-1-2s)(2r-1)+1)\%((2r-1)^2)$$

$$u_{r,s} = ((2rs)^2-4rs(s^2+s-1)+2(s-1)^2(s+1)-1)\%(((2r-1)s+1)^2)\ .$$

Case 111.

$$f(x,r,s) = ((2r-1)^2x+t_{r,s})(((2r-1)(2s-1)+2)^2x+u_{r,s})$$

$$t_{1,u} = 1;\ \text{if } r>1 \text{ then }\ t_{r,s} = (r+s-2rs)\%((2r-1)^2)$$

$$u_{r,s} = ((4s-2)^2r^2-(2s-1)((2s+1)^2-12)r+(s-1)((2s-1)^2-8))\%(((2r-1)(2s-1)+2)^2)\ .$$

Just as in the case of 4-pals on p.19, coefficients of Bézout's identity are not invoked explicitly, still, they have appeared. What makes it here even more surprising is the possibility to obtain them as singletons, not in couples.

6. Negative Pell equation

Looking for even-length pal polynomials one is forced to meet this topic. Very naive questions are: when dealing with the equation $b^2+1=v^2c$ how to delimit positions (in the triangle in p.3) of those c's that make it solvable? And once the description is given why in these regions there are so many odd-lenth pals?

The symbol **Pri1** will denote the set of primes p with $p\%4=1$ and **Pri3** the set of primes p with $p\%4=3$.

The question of positions is easy to answer. The cell c of any even-length pal must satisfy $c\%4=1$ or $c\%4=2$ and no prime $p\in$ **Pri3** can divide c. So, in the triangle on the next page one marks the acceptable regions with circles (for values of the polynomial x^2+1!) and 2x2 'green border boxes' and then in the boxes one excludes c's that are multiples of primes from **Pri3**. And then there arises hard question: why some of positions indicate odd-length pals?

Substitution of the question with an answer: *'if c has odd-length pal then the equation $b^2+1=v^2c$ has no solutions in $\mathbb{N}$'* is a consolation prize but it brings little satisfaction. Here are in red ovals the no-solution cells of the first 20 lines of the triangle.

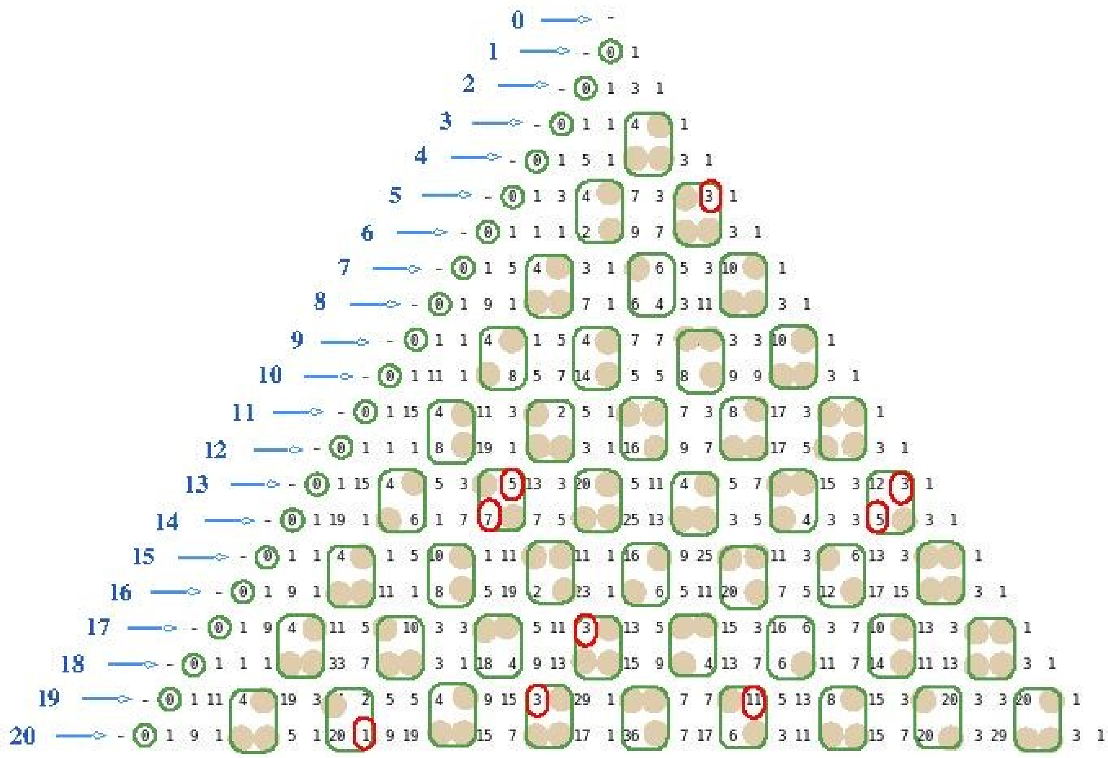

FIGURE 4 – Cells for even-length pals

Before getting more involved in the question of 'red oval cells', that is: of numbers that bring negative answer to negative Pell equation, the positive answer may strike a chord with those who study the subject:

Every solution of negative Pell equation comes from polynomials constructed on pp.14-15: for a proposed equation $B^2+1=V^2D$ verify if $cf(\sqrt{D})$ has even-length pal; if the answer is 'yes', construct its polynomial $f(x)=v^2x^2+2bx+c$, put $V=v$, and $B=\sqrt{v^2D-1}$.

6.1. On cases with no solution

As far as negative Pell equations without solution are concerned, 3-pals located in the cells in the second se-diagonal in Figure 5 are already understood: for $c=34$ the pal is $\ll 1|4|$, for $c=194$ it is $\ll 1|12|$, in both cases the polynomials are obtained as $\hat{f}$ for some 1-pal polynomials f. And inspecting the cases with no solution the paper of sw-diagonals with numbers k^2 ($k=3,4,5$ in Figure 5), is impressive; for example it is the list of cases with odd-length pals that lie in rows with numbers less than 100:

k	row numbers
3	13,14,28,29,35,41,43,44,49,53,64,68,73,77,83,85,86,92,94
4	17,19,23,27,39,45,47,59,61,71,73,77,83,97
5	14,19,21,29,37,38,43,44,48,59,61,66,69,71,83,84,88,91,92,93,99
6	37,41,43,47,53,61,67,89,97
7	26,29,34,36,48,51,57,62,64,66,69,73,74,79,82,86,90,99
8	71,85
9	55,68,70,82,89,97

There are also 5 cases of 3-pals in the second se-diagonal (row numbers 5,13,21,25,29) and 5 astray cases of $22''21, 22''30, 23''33, 26''30, 28''18$ (all of them even numbers).

However, there is a simple method to produce numerous examples that do not lie in sw-diagonals with numbers k^2. It consists in starting with a number that has no solution for negative Pell equation and then generating the polynomial for its pal. Three cases are presented here, for $2{\cdot}17$ (the very first of the suite), $13{\cdot}17$ and $5{\cdot}41$. Their pals and pal polynomials, respectively, are:

(i) $\ll 1|4|$ $\quad f(x) = (3^2x+8)(x+1)$,
(ii) $\ll 1,6|2|$ $\quad f(x) = (8^2x+17)(7^2x+13)$,
(iii) $\ll 3,6,1|4|$ $\quad f(x) = (63^2x+41)(22^2x+5)$.

Analysis of values of these polynomials when $0 \leqslant x \leqslant 500$ shows that the following arguments x give numbers that lie in 'green border boxes':

(i) for $x = 25, 49, 121, 177, 193, 249, 289, 313, 337, 361, 393, 457$ the values of $f(x)$ do not lie on sw-diagonals with numbers k^2;

it occurs for arguments $x = 1, 9, 33, 57, 105, 145, 217, 273, 369, 441$, here the values of $f(x)$ are equal to $5''3^2, 29''7^2, 101''13^2, 173''17^2, 317''23^2, 437''27^2, 653''33^2, 821''37^2, 1109''43^2, 1325''47^2$;

(ii) $x = 12, 21, 24, 45, 72, 84, 96, 108, 132, 141, 189, 204, 216, 237, 240, 276, 285, 309, 324, 336, 357, 372, 381, 405, 420, 453, 492$ – none of the values is on sw-diagonals with numbers k^2;

it occurs for argument $x=0$, as $13\cdot17=14''5^2$ and for $x=408$, as $f(408)=22862''199^2$;

(iii) $x=5,8,12,17,24,32,33,36,56,60,84,104,108,116,120,129,132,137,140,161,188,204,216,236,260,272,284,288,305,308,320,360,368,369,377,384,396,404,417,420,441,449,452,465,480,489,497,500$ – again, none of the values is on sw-diagonals with numbers k^2;

it occurs for argument $x=0$, as $5\cdot41=14''3^2$.

The difficulty in studying which numbers have no solution for negative Pell equation resides in shortage of methods to deal with the question. Legendre symbol $\left(\frac{p}{q}\right)$ for quadratic residues is useful to some degree (for example $\left(\frac{97}{113}\right)=-1$, so for $97\cdot113$ there is a solution) but $\left(\frac{5}{13\cdot17\cdot29\cdot41}\right)=1$ may mean that $(-1)^{2k}=1$. It is possible to skirt it around and get $\left(\frac{5\cdot13\cdot17}{29\cdot41}\right)=-1$ (in fact, $\sqrt{5\cdot13\cdot17\cdot29\cdot41}$ has 10-pal $\ll4,3,572,1,4|$) but such artisanal methods will not bring full answers.

Chasing them (which ones? how many?) seems to set aside a more fundamental question: *why should one look for them*?

The possible answer is that they may hide an agebraic structure which could be relevant in number theory. The '**OL** set' (of numbers with odd-length pals) bears some resemblance to the ideal in ring theory. Product of any $n\in\mathbb{N}$ with 4 or with prime $p\in$**Pri3** is a number of the set. But it is no black hole – some products of primes from **Pri1** can come and go. An example of $5\cdot61$ (with 3-pal $\ll2|6|$) and $13\cdot29$ (with 3-pal $\ll2|2|$) shows it: $5\cdot13\cdot29\cdot61$ has 16-pal $\ll10,1,1,2,7,1,41,1|$.

It is risky to base intuitions on few examples but they sometimes activate the imagination. One of them may serve as a representative of their kin. (Here $l(n)$ will denote the length of pal of $cf(\sqrt{n})$.)

All products of three primes q from **Pri1**, namely $53,89,97$, land in **OL**:

$l(53\cdot89)=39,\quad l(53\cdot97)=23,\quad l(89\cdot97)=39,\quad l(53\cdot89\cdot97)=35.$

Then there comes the number 41,

$l(41\cdot53)=6,\quad l(41\cdot89)=42,\quad l(41\cdot97)=20,$

and makes all the products leave **OL**:

$l(41\cdot53\cdot89)=12,\quad l(41\cdot53\cdot97)=232,\quad l(41\cdot89\cdot97)=114,\quad l(41\cdot53\cdot89\cdot97)=332$.

One would wish to know the type of pal for every product of any subset of elements in $A = \{2\} \cup \mathbf{Pri1}$ – or, to put in more formalized terms, if $F^A \subset 2^A$ denotes the family of finite subsets of A, to know the values of the function

$$P : F^A \longrightarrow \{0,1\}$$

which attributes 0 to subset $\{q_1, \ldots q_k\} \in F^A$ if the pal of $cf(\sqrt{q_1 \cdot \cdots \cdot q_k})$ has even length, and 1 otherwise. Only the trivial fact is known about this function: unitary subsets give value 0.

6.2. No easy criterion for two factors

There is no simple method in view to find the type of pal for products of two primes from **Pri1** or product of one of them with number 2. If Legendre symbol of such a pair is -1 then the product leads to an even-length pal but the contrary result does not guarantee that odd-lenth pal will be received. The most obvious cases are when these products give number of the form k^2+1; for example, 5 is quadratic residue modulo 29, as $11^2 \,\%29 = 5$ but $5 \cdot 29 = 12^2+1$, so that $cf(\sqrt{5 \cdot 29})$ has empty pal. Still, this situation by no means is exceptional; numerical data depend on fixing one of factors and deciding how far the other factor will run but invariably results tend to show that between 20% and 40% cases of the batch lead to even-length pals.

Examining pals of two prime factors follows a background conviction that a case of two is easier than a case with many. It may be true – or not. Perhaps another way, forming clusters of many primes where product of every subset gives even-length pal, offers more hues to understand the general rule. Still, it would envolve a lot of experiments.

6.3. A class of promising examples

The way values of $f(x)$ constructed for odd-length pal change pals when multiplied by n^2 is left as one of problems in the section 8.

(The product $n^2 f(x)$ is not a pal polynomial any more and its values constitute a subsequence of a new pal polynomial.)

The situation is different for products of any even-length pal polynomial $f(x)$.

Multiplying it with n^2 for any $n \in \mathbb{N}^*$ (and the product is denoted by $n;x$ when it is clear which $f(x)$ is used) reveals regularities that do not materialize when products pq for $p,q \in \mathbf{Pri1}$ are examined.

Informations produced in this way relate to odd-length pals if n is even (the obtained products are multiples of 4, so they gives odd-length pals) and it will be also true when n has any divisor from **Pri3**. Still, such results deserve a closer look, so at least two cases will be presented here. First, multiplying $f(x)=25x^2+14x+2$ by 11^2 one gets

$x\%11$	0	1	2	3	4	5
$l(n;x)$	15	55	45	45	55	5
$l(n;x)$	15	55	49	45	55	
$x\%11$	-1	-2	-3	-4	-5	

– the shortest pal occurs for $x=11k+5,\, k \in \mathbb{N}$ and it is $\ll 2,2|10k+4|$.

The same polynomial multiplied by 26^2 gives the following lengths of pals:

$x\%13$	1	2	3	4	5	6	7
$l(n;x)$	73	129	25	21	73	57	5
$l(n;x)$	61	129	25	25	69	73	
$x\%13$	0	-1	-2	-3	-4	-5	

Here the shortest pal occurs for $x=13k+7,\, k \in \mathbb{N}$ and it is $\ll 2,2|5k+2|$.

The regularity described in these tables starts at that value of argument x when for the first time there is $n^2 > f(x)$.

What informations on pals can be received when n itself belongs to **Pri1** (or it is a product of its elements) and $f(x)$ is constructed for an even-length pal? The procedure applied to the first possible example of $n^2 \cdot f(x) = 5^2 \cdot (4x^2+1)$ shows the results in the following series of pals:

x	pal	$l(n;x)$
$5k$	$\ll 5k \gg$	1
$5k+1$	$\ll 4k, 1, 4, 4k, 1, 1\vert$	12
$5k+2$	$\ll 4k+1, 1, 1\vert$	6
$5k+3$	$\ll 4k+2, 2\vert$	4
$5k+4$	$\ll 4k+3, 4, 1, 4k+2, 2\vert$	10

For the case of $17^2{\cdot}(4x^2{+}1)$ the regularity starts at $x{=}9$ (when $x<9$ the factor $4x^2{+}1$ is smaller than 17^2) and that is how lengths of pals depend on $x\%17$:

$x\%17$	0	1	2	3	4	5	6	7	8
$l(n;x)$	1	4	68	30	30	31	11	32	27
$l(n;x)$		6	66	36	28	27	11	30	23
$x\%17$		-1	-2	-3	-4	-5	-6	-7	-8

The pals form 10 series of even length and 7 of odd one. For example, the series $x=17k+7$ gives 32-pals $\ll 4k+1,1,1,1,5,4k+1,2,8,4k+1,1,1,7,1,4k,1,3|$ and $x{=}17k{+}1$ gives 4-pals $\ll 4k,4|$.

If the polynomial $13^2x^2+140x+29$ is multiplied by 41^2, one gets

$x\%41$	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
$l(n;x)$	40	150	142	131	127	69	48	146	40	34	290	134	65	131	34	16	151	27	150	150	9
$l(n;x)$	40	154	146	139	131	69	40	150	48	26	290	142	69	139	26	16	147	27	154	154	
$x\%41$	6	5	4	3	2	1	0	-1	-2	-3	-4	-5	-6	-7	-8	-9	-10	-11	-12	-13	

The shortest pals here are with $x=27{+}41k$, they are $\ll 1,3,2,1|16+26k|$. By the way, for $x{>}3$ *all* pals here start with $'1,3,2,1'$.

The next example uses again the polynomial $4x^2+1$ multiplied by 37^2.

$x\%37$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$l(n;x)$	1	80	90	176	25	42	81	81	88	84	70	34	86	85	90	94	36	6	84
$l(n;x)$		82	92	174	25	36	93	85	86	82	72	36	88	73	80	80	26	4	86
$x\%37$		-1	-2	-3	-4	-5	-6	-7	-8	-9	-10	-11	-12	-13	-14	-15	-16	-17	-18

Here the shortest pals are $\ll 1{+}4k,1,5|$ obtained with $x=17{+}37k$ and $\ll 2{+}4k,6|$ coming for $x=20{+}37k$.

One more example, with $f(x)=53^2x^2+1000x+89$ multiplied by 13^2:

$x\%13$	1	2	3	4	5	6	7
$l(n;x)$	56	94	24	33	44	44	13
$l(n;x)$	48	90	20	49	52	56	
$x\%13$	0	-1	-2	-3	-4	-5	

There are two important points in these data: first of all, the regularities exist. And next, they have connection with some divisibility relations.

7. Pal polynomials and prime numbers

Primes from **Pri3** are solitary. Each of them is the initial value of its pal polinomial, either as $p = f(1)$ (if $p = k^2+2$ and $f(x) = (k^2x+2)x$) or as $p = f(0)$ (when $f(x) = (r^2x+p)(s^2x+1)$ with odd r, s). The discriminant of $f(x)$ is always 4.

Primes from **Pri1** are gregarious and flock together in their polynomials. An eloquent sample with data of polynomials of three simplest even-length pals: $\ll \gg$, $\ll 2|$ and $\ll 1, 1|$ is in the next table.

The quantities of prime values of these polynomials, with jumps of limits on variable x from 2^7 to 2^{25}, are compared with the those of prime numbers in $\mathbb{N}$.
$p(k)$ - number of prime numbers
$a(k)$ - number of prime values of $x^2 + 1$
$b(k)$ - number of prime values of $25x^2 + 14x + 2$
$c(k)$ - number of prime values of $25x^2 + 36x + 13$

The quotients are truncated to 10^{-4}

$x < 2^k$	$p(k)$	$a(k)$	a/p	$b(k)$	b/p	$c(k)$	c/p
$x < 2^7$	31	23	0.7419	22	0.7096	24	0.7741
$x < 2^8$	54	41	0.7592	41	0.7592	36	0.6666
$x < 2^9$	97	69	0.7113	77	0.7938	66	0.6804
$x < 2^{10}$	172	113	0.6569	134	0.7790	128	0.7441
$x < 2^{11}$	309	211	0.6828	235	0.7605	234	0.7572
$x < 2^{12}$	564	392	0.6950	443	0.7854	430	0.7624
$x < 2^{13}$	1028	712	0.6926	813	0.7908	794	0.7723
$x < 2^{14}$	1900	1300	0.6842	1509	0.7942	1484	0.7810
$x < 2^{15}$	3512	2458	0.6998	2778	0.7910	2751	0.7833
$x < 2^{16}$	6542	4614	0.7052	5191	0.7934	5205	0.7956
$x < 2^{17}$	12251	8417	0.6870	9733	0.7944	9794	0.7994
$x < 2^{18}$	23000	15866	0.6898	18508	0.8046	18396	0.7998
$x < 2^{19}$	43390	29842	0.6877	35065	0.8081	34987	0.8063
$x < 2^{20}$	82025	56533	0.6892	66627	0.8122	66675	0.8128
$x < 2^{21}$	155611	106786	0.6862	127292	0.8180	127402	0.8187
$x < 2^{22}$	295947	203024	0.6860	243200	0.8217	243566	0.8230
$x < 2^{23}$	564163	387307	0.6865	465375	0.8248	465790	0.8256
$x < 2^{24}$	1077871	739670	0.6862	893816	0.8292	893780	0.8292
$x < 2^{25}$	2063689	1416634	0.6864	1718947	0.8329	1718295	0.8326

Other polynomials do not give so high quotients but the flux of primes in numerous experiments suggests that the famous question of Edmund Landau on primes in the polynomial $x^2 + 1$ might be so reformulated:

Let $f(x)$ be pal polynomial for a $2k$-pal. Is it posssible that in its set of values for $x \in \mathbb{N}$ the subset of primes is finite?

8. Loose ends

8.1. Two easy questions

What about the uniqueness of pal polynomials? The dissolution of the question was promised on p.9. Well, if a pal is given and its polynomial $f(x)$ is constructed, its descent index v (equal r^2+s^2 for even-length pals and rs for odd-length ones) indicates the distance between rows with consecutive values of the polynomial.

Were there a putative number with the same pal between two such rows, with distance $v' < v$ to one of them, one would create another polynomial with descent index v'. But admissible descent indices are used by distinct pals, so for that number in question the continued fraction of its square root would not be uniquely determined.

Neither there is a chance to appear two numbers with the same pal in a certain row (it is the string, not its length, considered). Here it would mean that $v' = v$ but then r, s would not be contested, so Bézout's identity would lead to the initial polynomial $f(x)$.

Another point that might need a clarification is the characterization of terms v and c from the equation $b^2 + 1 = v^2 c$ on p.24.

The possible values of $(b^2+1)\%4$ are 1 and 2, so v cannot be even. As it is obtained from a Pythagorean triple as a sum of two squares, it has no factors from **Pri3**. Neither c has them because it is equal to t^2+u^2, as the algorithm on p.15 shows.

8.2. The realm of problems

A stimulating phrase of mathematics *'it is an open question'* is the euphemism for *'I don't understand what is going on here'*. Some of open questions will be listed now, others will probably appear without inducement.

(*) Let the story of sw-diagonal number 49 be the first of them. Probably it is of no consequence, mysterious things need not be important. The range of (odd-numbered) rows analyzed here is from 25 to 1001.

Pals of products $2p$ with $p \in \mathbf{Pri1}$ have quite different central part for odd and even lengths. In odd-length pals (excluding cases of 1-pals, like for $n = 2504$), the central coefficients is equal to the row number of $2p$ (if it is even) or the row number minus 1 (if it is odd). And for even-length pals the central pair is 1's in nearly half of the cases; the others, that do not lie on 49^{th} sw-diagonal, keep the quotient '*central term divided by row number*' below 0.25.

But products with even-length pals, if the are on this diagonal – be what may the length of the pal (varying from 6 to 970) or Legendre symbol of the pair $2, p$ – either have 1 in their central pair (13 cases of total 79) or break this limit (the remaining 66 cases). There the quotient '*central term divided by row number*' stays close to 0.285.

(*) The next topic is related to numbers that appear in odd-length pals. It is quite often that the number of the row (or this number minus one) appears as the central element of the pal. This never happens with even-length pals. But other elements with high quotient when divided by the row number show high regularity – basically the quotients come close to either 1/2 or 2/3. Characterization of numbers with these properties might be worthwhile.

(*) Multiplication of pal polynomials by perfect squares brings many surprises. It may be illustrated by $f(x) = (9^2x + 53)(28^2x + 513)$ that comes for 5-pal $\ll 1, 8|6|$. It does not present any affinity to number 7, in fact, its consecutive values modulo 7 give the cycle $1, 2, 3, 4, 5, 6, 0$, and their row numbers are always 3 modulo 7. Still, when multiplied by squares of $9, 18, 36, 72$, its values become elements of polynomials of 1-pals $\ll 56 \gg$, $\ll 28 \gg$, $\ll 14 \gg$ and $\ll 7 \gg$, respectively. A very original way to confirm that certain numbers of rows are multiples of 7.

It is a modest gate opening path to some spectacular connections among pals and their polynomials.

(*) The last example shows that multiplying n^2 and the polynomial formed for odd-length pal gives results totally different than multiplication by polynomials of even-length pals, discussed in the previous section: obtained pals do *not* depend on x. The polynomial $f(x) = (6^2x + 1)(19^2x + 10)$ shows what new possibilities come into play:

$$\begin{array}{rl}
p(x) & \lll 6|6| \\
2^2p(x) & \lll 3|12| \\
3^2p(x) & \lll 2|18| \\
6^2p(x) & \lll 1|36| \\
4^2p(x) & \lll 1,1,1|5| \\
9^2p(x) & \lll 2,5,1,4,1,5{+}228x,2|56|
\end{array}$$

Clearly, the last line is a new quality, and it will be discussed soon. Here it may be useful to stress once more that the products on the left are *not* pal polynomials of pals on the right, they only jump in orderly manner through the new pal polynomials. There is a carnival of unexpected results taking $g(x){=}(9^2x{+}28)(28^2x{+}271)$ together with $\hat{g}(x)$ (that is, letting x run also through negative integers) and multiplying it by squares of $k < 100$; at least 24 obtained series are truly impressive.

(*) The line with results on $9^2p(x)$ leads to the notion of the **linking polynomial**. (In fact, these polynomials have already appeared in tables of previous section as parameters inside of pals.) It is not a pal polynomial but some way of grouping pal polynomials, possibly of distinct lengths. A cursory glance at the triangle on p.3 suggests the simplest example, using the second se-diagonal. One can see that $h(y){=}y^2{+}6y{+}7$ runs for consecutive arguments $y \geqslant 0$ through pals $\lll 1|y|$.

Less obvious example can be found inspecting 4^{th} se-diagonal of the triangle – its even-numbered rows have 5-pals $\lll 1, y+1|2|$ and they lead to the polynomial $(2y{+}4)^2{+}(4y{+}5){=}(2y{+}3)(2y{+}7)$.

Joining numbers by their pals seems fruitful. Joining pals by their similarilies may turn out to be even more fruitful.

(*) The path indicated by traffic sign 'study the size of quotient $l(c)/\lfloor\sqrt{c}\rfloor$' is well trodden but does not seem to lead anywhere. And it might be as well separated in two independent questions, for odd-length and even-length pals, as the structure of c is different, once it is tu, otherwise it is t^2+u^2, where t,u are coefficients of Bézout's identity for some r,s.

It seems that the search of limits (or their lack) for $v/\lfloor\sqrt{c}\rfloor$ might be more interesting. The examples of $c=379$ and $c=541$ in p.11 demonstrate that in both cases, $v=rs$ and $v=r^2+s^2$, may be bigger than one could ask for.

(*) If such limits exist, they might indicate whether it is reasonable to try another approach to find pal polynomial for some products pq *not knowing* its pal.

On the sunny side of the street there is a pair $13,17$. Suppose that an odd pal is to be expected, and one looks for a polynomial of the form $(r^2x+13)(s^2x+17)$ with $\det\left(\begin{smallmatrix} r^2 & 13 \\ s^2 & 17\end{smallmatrix}\right)=\pm1$. So, the question is: *for what pair of perfect squares the numbers* $13,17$ *are coefficients of Bézout's identity?* As $\det\left(\begin{smallmatrix} 10 & 13 \\ 13 & 17\end{smallmatrix}\right)=1$, so is $\det\left(\begin{smallmatrix} 13k+10 & 13 \\ 17k+13 & 17\end{smallmatrix}\right)$ for any $k\in\mathbb{N}$. Can its first column be equal $\left(\begin{smallmatrix} r^2 \\ s^2\end{smallmatrix}\right)$? Or should matrix $\left(\begin{smallmatrix} 3 & 13 \\ 4 & 17\end{smallmatrix}\right)$ with $\det=-1$ be used here? In this case the equation would be $\left(\begin{smallmatrix} r^2 \\ s^2\end{smallmatrix}\right)=\left(\begin{smallmatrix} 13k+3 \\ 17k+4\end{smallmatrix}\right)$.

Luckily the first variant gives the desired solution for $k=3$ and the odd-length polynomial $(7^2x+13)(8^2x+17)$ is obtained, although the information on $l(13\cdot17)$ did not surface.

However, the same exercise with the pair $29,53$ shows how hard it can be. It boils down to finding an algorithm for solving equations of the form $a^2m+b^2n=1$.

A candidate for such an algorithm is applicable if the said limit (say, number z) is given at the start. Then in the worst case, having executed $16z$ failed tentatives, the algorithm will leave with information that the sought pair a^2,b^2 does not exist. If z were not given, there would be no way to know when the search should be dropped. (Presenting this algorithm is useless as long as the existence of the limit z is not established.)

Direct calculation is not an option. For the pair $29,53$ the solution is obtained with $k=24,424,310$ and the polynomial is $f(x)=(26614^2x+29)(35979^2+53)$.

(*) Another question is connected with the example cited in subsection 5.2. The polynomial $(5^2x+1)(18^2x+13)$ starts a series with $(5^2x+1)(43^2x+74)$ in the following step. In the next ones the second factor has coefficient $(18+25k)^2$ at x and the consecutive free terms are values of the polynomial $g(x)=5^2x^2+36x+13$. At first sight it looks like a misunderstanding – the 4-pal $\lessdot 1,1|$ leading to values $f(0)$ of polynomials with 9-pals! Still, it is a nice example how illusive roots of consecutive $f(x)$'s are chosen among values of polynomial $g(x)$, and $g(x)$ itself has its staple root equal to 13. Every $2k$-pal can be used to form the series of $(4k+1)$-pals in an analogous way. Hence the presence of primes $p\in$**Pri1** in odd-length polynomials that have form $(r^2+1)(s^2+p)$. Seeing these free coefficients $1,p$ one knows that the polynomial's root is illusive. And for other polynomials $f(x)$ with odd-length pals, is it possible to create a more general criterion saying what type of root is $f(0)$?

9. Entry and exit of the pronoun 'I'

The 'Pol4Pal Calculator', computer implementation of algorithms from the article, has been written in JavaScript by Igor da Silva Solecki. It can be accessed at `http://www.andsol.org/2020/pol4pal.html`. The code of its variant in Python will be also included in the same directory.

If you get interesting results related to this article and tell me about them by e-mail, it will be my pleasure to put a suitable link to a file on feedback.

Using the word 'I' in mathematical text is not helpful. The tradition has it that author stays in the background. I reckon that it makes sense. Mathematics is a collective activity, its words keep memory of accords, edits and modifications. They are common values. Someone giving the term 'matrix' a personal meaning would be a laughing stock – unless it were a part of some well founded revolution. Therefore, much attention is needed with traditional values.

So I have better explain why there is no bibliography. Not 'the author explains' but I do. Using passive voice and impersonal language has been my duty in previous sections; here it would be avoiding the responsability. Yes, I look with suspicion at writing with no references. So will do others who decide to read me.

In the last hundred years making bibliography became an art. Very impressive, like the caligraphy in Byzantine period. It gives weight to the article. It has weight. Or it had.

It may be anything – a proof of erudition, a help for researcher in the field of history of ideas, a conventional nod to colleagues working in the area. Less and less it serves as a help for a reader. A small group of readers can find the cited volumes in their institutions; most of them can ascertain on the Internet that the article can be acquired for some absurd sum. So they go to any source that uses similar terms. I do not evaluate the facts, I only state them.

It would be my undeniable duty to mention authors of recent results that I have used here. Well, there are none. In fact, among impulses that directed me to continued fractions were articles of V.I. Arnold ('Arithmetics of binary quadratic forms, symmetry of their continued fractions and geometry of their de Sitter word' published in 2003 by Sociedade Brasileira de Matemática, and 'Lengths of periods of continued fractions of square roots of integers' of 2009 that he published in his journal 'Functional Analysis and Other Mathematics'). Not what they contained but what they did not. I thought it strange that there was such a great mind, so sophisticated apparatus and with no reasonable algebra results sliding down to tables with some statistics. With my full respect for statistics, I see it as a company, not substitute for algebraic or geometric vision.

There are no acknowledgements, either, of grants, as there were none. Perhaps because I did not apply for any.

But I have not been on my own in my work. Obviously, I am pleased that 250 years after Joseph Louis Lagrange had announced his results I could add a useful contribution to the subject. Still, there is no shade of doubt that I owe my success to authors of most varied computer programs that I was using in my work. Too many programs to mention all of them, so I name here at least some of those that I wouldn't do without: grep, LaTeX, Linux, OEIS, pari-gp, Python, sed, vim.

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